Pergamon

www.elsevier.com/locate/jappmathmech

PII: S0021–8928(00)00131-3

J. Appl. Maths Mechs, Vol. 64, No. 6, pp. 993–1004, 2000

0021–8928/00/\$—see front matter

THE OPTIMUM SHAPE OF A BENDING BEAM†

A. S. BRATUS' and V. P. POSVYANSKII

Moscow

(*Received* 16 *December* 1999)

The problem of minimizing the static deflection of an elastic beam of variable cross-section and fixed volume in the case of free supported and rigidly clamped ends is considered. In the first case it is proved that the solutions obtained earlier, based on the necessary conditions for an extremum, satisfy the sufficient conditions. In the case of clamped ends, which is of the most interest from the point of view of applications, it is proved that the optimum solutions must necessarily have points inside the solution range in which the distribution of the beam thicknesses degenerates to zero ("internal hinges"). A qualitative, analytical and numerical analysis of this phenomenon is given. In particular, in the case of clamped ends for a class of point loads, analytical solutions for which the beam splits into two cantilevers are obtained.

1. FORMULATION OF THE PROBLEM

Consider the problem of the static deflection of an elastic beam of variable cross-section

$$Au(x) = (EI(x)u''(x))'' = g(x), \quad x \in (-l, l), \quad l > 0$$

Here $u(x)$ is the deflection, $I(x)$ is the moment of inertia, E is Young's modulus and $g(x)$ is the distribution of the values of the transverse force applied to the beam. We will investigate two conditions at the ends: free support and rigid clamping. Without loss of generality we can assume that $l = 1$ and

$$EI(x) = h^p(x)$$

where h is the cross-section area and p is a real parameter, which takes the values 1, 2 or 3 [1, 3]. Note that, from more general considerations, it is also legitimate to investigate this problem for any other real values of $p \neq 0$.

We finally arrive at the following boundary-value problems (1.1), (1.2) or (1.1), (1.3)

$$A(h)u(x) = (h^p(x)u''(x))'' = f(x) \tag{1.1}$$

$$u(1) = u(-1) = 0, \quad (h^p(x)u'')_{x=1} = (h^p(x)u'')_{x=-1} = 0 \tag{1.2}$$

$$u(1) = u(-1) = 0, \quad u'(1) = u'(-1) = 0 \tag{1.3}$$

Here we will denote by Q the set of bounded measured functions $h(x)$, which satisfy the conditions

$$h(x) \geq 0, \quad \int_{-1}^{1} h(x)dx = 1, \quad \int_{-1}^{1} h^{-p}(x)dx < +\infty \tag{1.4}$$

It is well known [3], that the last condition is sufficient for the operator $A(h)$ to be positive-definite.

At the ends of the section the order of degeneracy may not exceed unity [3]. It follows from this condition that when $p > 0$ the order of degeneracy of the functions to zero inside the section must not exceed $1/p$, i.e. if for a certain x_0 the equation $h(x_0) = 0$ is satisfied, we have

$$|h(x)| \leq K|x - x_0|^{\gamma}, \quad K = \text{const}, \quad \text{as} \quad x \to x_0; \quad \gamma < 1/p$$

For any function $h \in Q$ and $p \in \mathbb{R}$ we will consider a space $W^2_{h,p}$ of functions which vanish at the ends of the section [−1, 1], with norm

$$\|u\|_{W^2_{h,p}} = \left(\int_{-1}^{1} (u''(x))^2 h^p(x)dx \right)^{1/2}$$

†*Prikl. Mat. Mekh.* Vol. 64, No. 6, pp. 1033–1045, 2000.

With these assumptions the operator $A(h)$ will be positive-definite on any functions $h(x) \in Q$ [3], where imbedding of the space $W^2_{h,p}$ into the space of continuous functions in the section $[-1, 1]$ will occur, i.e. a constant $\delta > 0$ exists such that [4]

$$\|u\|_{W^2_{h,p}} \geqslant \delta \max_{x\in[-1,1]} |u(x)|, \quad \forall u \in W^2_{h,p}$$

We will denote by W′ the space dual to the space $W^2_{h,p}$. We will further assume that $f \in W'$. This, in particular, implies that the function $f(x)$ can be a Dirac delta function (a point force).

The solution of each of boundary-value problems (1.1), (1.2) and (1.1), (1.3) is understood in the weak sense: i.e. $u(x)$ is a solution if for any function $w \in W^2_{h,p}$ the following integral identity is satisfied

$$(A(h)u, w) = (f, w)$$

Here and henceforth (,) is a scalar product in space L_2.

Consider the functional $\Phi(h)$: $Q \to \mathbb{R}$ of the form

$$\Phi(h) = (A(h)u, u) = (f, u) \tag{1.5}$$

where $u(x)$ is the solution of problems (1.1), (1.2) and (1.1), (1.3). The functional $\Phi(h)$ expresses the potential energy of the deformation of the beam due to the action of the distributed load $f(x)$. Using boundary conditions (1.2) or (1.3) it can be shown that

$$\|u\|^2_{W^2_{h,p}} = \Phi(h)$$

On the other hand, it follows from the imbedding inequality that for any p and $h \in Q$ the absolute value of the deflection of the beam has an upper limit $\delta^{-1}\Phi(h)$.

If $f(x)$ is a delta function (a point force), the functional $\Phi(h)$ is exactly equal to the deflection of the beam at its point of application. Consequently, functional (1.5) can be regarded as a fairly adequate measure of the deflection of the beam.

It is natural to consider the following extremal problem. For a specified value of the parameter p and a specified distribution of the transverse forces $f(x)$ it is required to obtain a function $h(x) \in Q$ such that functional (1.5) reaches its minimum value. In other words, we seek such a value of the distribution of the cross-section areas of a beam of specified volume for which the measure of the deflection of the beam (in the sense mentioned above) under the action of a transverse load is a minimum.

Solutions of this problem were obtained previously [5–7] for $p = 1, 2, 3$ and different forms of boundary conditions, on the basis of the necessary conditions for an extremum. The purpose of the present paper is to verify that the necessary conditions for a second-order extremum (close to sufficient) are satisfied for the previously obtained solutions.

2. THE AUXILIARY EIGENVALUE PROBLEM

For what follows it will be useful to investigate the following eigenvalue problem

$$A(h)\, u(x) = -\lambda u''(x), \quad x \in (-1, 1) \tag{2.1}$$

with boundary conditions (1.2) or (1.3).

Assuming $y = h^p(x)u''(x)$, we can write problem (2.1) in the form

$$y''(x) + \lambda u''(x) = 0$$

Integrating this equation, taking conditions (1.2) or (1.3) into account, we arrive at the equation

$$y''(x) + \lambda h^{-p}(x)\, y(x) = 0 \tag{2.2}$$

with either of the following boundary conditions

$$y(-1) = y(1) = 0 \tag{2.3}$$

$$y(1) = y(-1) + 2y'(-1), \ \ y'(1) = y'(-1) \tag{2.4}$$

which correspond to conditions (1.2) and (1.3).

If $\lambda \neq 0$, there is a one-to-one correspondence between the solutions of problems (2.1), (1.2) or (2.1), (1.3) and the solutions of problems (2.2), (2.3) or (2.2), (2.4), respectively. It is easy to verify that problem (2.2), (2.3) has no zero eigenvalues, which cannot be said of problem (2.2), (2.4), which has a multiple zero eigenvalue $\lambda_1^0 = \lambda_2^0 = 0$ and two corresponding eigenfunctions $y_0^1(x) = c_1$ and $y_2^0(x) = c_2 + c_3x$, where $c_i = \text{const}$ ($i = 1, 2, 3$).

We know [3], that the systems of eigenfunctions $\{y_i(x)\}$ of problem (2.2), (2.3) or (2.2), (2.4) are complete in the space of functions $L^2_{h,p}$ with norm

$$\|y\|_{L^2_{h,p}} = \left(\int_{-1}^{1} h^{-p}(x)y^2(x)dx\right)^{1/2} \tag{2.5}$$

and the corresponding eigenvalues satisfy the inequalities

$$0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_n \leqslant \ldots$$

In the case of problem (2.2), (2.3) all the eigenvalues are simple, while in the case of problem (2.2), (2.4) double eigenvalues may arise. The eigenfunctions can be chosen so that the following normalization conditions are satisfied

$$\int_{-1}^{1} h^{-p}(x)y_i(x)y_j(x)dx = \delta_{ij} \tag{2.6}$$

where δ_{ij} is the Kronecker delta. In order that these conditions should also be satisfied for the functions $y_0^1 = c_1, y_2^0 = c_2 + c_3x$, corresponding to the double zero eigenvalue (problem (2.2), (2.4)) it is necessary to choose the constants c_i ($i = 1, 2, 3$) so that

$$c_1 = m_1^{-1/2}, \ \ c_2 = -m_1^{-1/2}m_2r^{-1}, \ \ c_3 = m_1^{1/2}r^{-1}$$

$$m_i = \int_{-1}^{1} x^{i-1}h^{-p}(x)dx, \ \ i = 1,2,3$$

$$r = m_1m_3 - m_2^2$$

When these conditions are satisfied we have the equations

$$\int_{-1}^{1} y_0^i(x)y_0^j(x)dx = \delta_{ij}, \ \ \int_{-1}^{1} y_0^i(x)y_k(x)dx = 0$$

$$i, j = 1, 2, \ \ k = 1, 2, 3, \ldots$$

Note that $r \neq 0$, since $m_2^2 < m_1m_3$ if $h_{(x)}^{-p} \neq \beta x$, $\beta = \text{const}$.

3. A VARIATIONAL ANALYSIS OF THE PROBLEM

Let us assume that $h_\varepsilon(x) = h(x) + \varepsilon\delta h(x)$, where $|\varepsilon| < \varepsilon_0$, ε_0 is a fairly small number and $h_\varepsilon(x)$ also belong to the set Q. It follows from conditions (1.4) that

$$\int_{-1}^{1} \delta h(x)dx = 0 \tag{3.1}$$

Moreover, the order of degeneracy of the function $\delta h(x)$ at points $x_0 \in (-1, 1)$ of degeneration of the function $h(x)$ ($h(x_0) = 0$) should not exceed $1/p$.

We will use the perturbation method of symmetrical boundary-value problems [8]. The function $u_\varepsilon(x)$, which is the solution of the perturbed boundary-value problems (1.1), (1.2) or (1.1), (1.3) with $h_\varepsilon = h(x) + \varepsilon\delta h(x)$ can be represented in the form of a series in integer powers of the small parameter

$$u_\varepsilon(x)=u(x)+\varepsilon v_1(x)+\varepsilon^2 v_2(x)+\varepsilon^3\psi_\varepsilon(x) \tag{3.2}$$

where $v_1(x)$ and $v_2(x)$ are certain functions from the space $W^2_{h,p}$, while $\psi_\varepsilon(x)$ is a bounded function in this space as $\varepsilon \to 0$. Substituting (3.2) into (1.1) and grouping terms of like powers of ε, we have

$$\begin{gathered}A(h)u(x)=f(x),\ A(h)v_1(x)=-A^1(h,\delta h)u(x)\\ A(h)v_2(x)=-A^1(h,\delta h)\,v_1(x)-A^2(h,\delta h)u(x)\\ A^i(h,\delta h)=\frac{1}{i!}\left(\frac{d^i}{d\varepsilon^i}A(h+\varepsilon\delta h)\right)\Bigg|_{\varepsilon=0},\quad i=1,2\end{gathered} \tag{3.3}$$

By virtue of the homogeneity of boundary conditions (1.2) and (1.3) the functions $v_1(x)$ and $v_2(x)$ also satisfy these boundary conditions.

If both sides of the second equation of (3.3) are multiplied scalarly in L_2 by the function $u(x)$, which is the solution of the first equation of (3.3), and we use the fact that the operator $A(h)$ is symmetrical, we obtain that

$$(v_1,f)=(v_1A(h)u)=(A(h)v_1,u)=-(A^{-1}(h,\delta h)u,u)$$

On the other hand, it follows from the definition of the functional $\Phi(h)$ in (1.4) and expansion (3.1) that

$$\Phi(h_\varepsilon)=(u,f)+\varepsilon(v_1,f)+\varepsilon^2(v_2,f)+O(\varepsilon^2) \tag{3.4}$$

Since

$$\begin{gathered}A^1u(x)=p(\psi(x),u''(x))''\\ \psi(x)=h^{p-1}(x)u''(x)\delta h(x)\end{gathered} \tag{3.5}$$

the expression for the first variation of the functional $\Phi(h)$ takes the form

$$\delta\Phi(h)=-p(h^{p-1}(u'')^2,\delta h)$$

It defines the first derivative with respect to direction (the Gâteaux derivative) of the functional $\Phi(h)$. Taking into account isoperimetric condition (3.1), the necessary conditions for an extremum can be written in the form

$$ph^{p-1}(x)(u''(x))^2=l_0^2,\quad l_0=\text{const} \tag{3.6}$$

Note that condition (3.6), generally speaking, does not impose any additional limitations on the order of degeneracy of the function $h(x)$, since at points where $h(x)=0$ the function $u''(x)$ can have a pole of the corresponding order, which ensures that Eq. (3.6) is satisfied. Here the value of the moment $h^p(x)u''(x)$ vanishes at this point.

Multiplying both sides of (3.6) scalarly in L_2 by the function $h(x)$ and taking the second equation in (1.4) into account, we obtain that $l_0^2=\Phi(h)$. We multiply both sides of the third equation in (3.3) scalarly by the function $u(x)$. It follows from relation (3.4) that

$$(A(h)v_2,u)=(v_2,A(h)u)=(v_2,f)$$

Hence

$$\delta^2\Phi(h)=(v_2,f)=-(A^1(h,\delta h)v_1,u)-(A^2(h,\delta h)\,u,u)$$

On the other hand, we have from the first equation of (3.3)

$$(A(h)v_1,v_1)=-(A^1(h,\delta h)\,u,v_1)=-(u,A^1(h,\delta h)v_1)$$

We finally obtain two equations, each of which determines the second Gâteaux derivative of the functional $\Phi(h)$

$$\delta^2\Phi(h)=\chi^p(h,u,\delta h)-p(h^{p-1}u''v_1''\delta h) \tag{3.7}$$

$$\delta^2\Phi(h)=\chi^p(h,u,\delta h)+(h^p v_1'',v_1'') \tag{3.8}$$

where

$$\chi^p(h,u,\delta h)=-p(p-1)(h^{p-2}(u'')^2,\delta h^2)/2$$

With the above assumptions regarding the order of degeneracy of the functions $h(x)\in Q$, formulae (3.7) and (3.8) have meaning for any permissible functions $\delta h(x)$.

It should be noted that, unlike the classical problems of the variational calculus, in this case the first and second Gâteaux derivatives (weak derivatives) are not strong derivatives (Frechet derivatives), if no additional smoothness conditions are imposed on the function $h(x)$. Hence, formulae (3.7) and (3.8) are suitable either for checking the necessary conditions for a second-order extremum or for proving that, at stationary points, a local extremum is not reached. The case when $\delta^2\Phi(h) > 0$ (<0) for any $h(x)\in Q$ and any functions $\delta h(x)$ is an exception. In this case we can assert that the functional $\Phi(h)$ is strictly convex (concave), and consequently, any local extremum will be global over the whole set Q. In particular, the following result, which does not depend on the form of the boundary conditions, holds.

Theorem 1. Suppose $f(x) \in W^1$. Then, when $0 < p < 1$, functional $\Phi(h)$ is strongly convex.

Proof. Consider expression (3.8) for $\delta^2\Phi(h)$. Since the second term is non-negative, we have the inequality

$$\delta^2\Phi(h) \geqslant \chi^p(h, u, \delta h)$$

The last expression is strictly positive when $0 < p < 1$ for any $\delta h \neq 0$.

4. THEOREM 2

Suppose $f(x) \in W'$ and that the necessary conditions for an extremum (3.6) are satisfied for a certain $h \in Q$. Then when $p \geqslant 1$ and $p < -1$ the functional $\Phi(h)$ can only reach a local minimum, and when $-1 < p < 0$ only a local maximum.

Proof. Consider the function $v_1(x)$, which is a solution of the second equation of (3.3). Since $v_1 \in W^2_{h,p}$, the function $v_1''h^p \in L^2_{h,p}$ where $L^2_{h,p}$ is the space of functions with norm defined by (2.5). Consequently, the function $v_1''h^p$ can be represented in the form of a linear combination of the functions $\{y_i(x)\}$ of boundary-value problem (2.2), (2.3), which form a complete system of functions in the space $L^2_{h,p}$ (everywhere henceforth the summation is carried out over i from $i = 1$ to $i = \infty$)

$$v_1''h^p(x)=\sum c_i y_i(x), \quad c_i=\text{const} \tag{4.1}$$

Substituting this expansion into the second equation of (3.3) and then multiplying both sides scalarly in L_2 by the function $y_1(x), y_2(x),\ldots$, we obtain

$$\sum c_i(y_i, y_j'')=-p(h^{p-1}u'', y_j''\delta h)$$

It follows from Eq. (2.2) and normalization conditions (2.6) that

$$(y_i, y_j'')=-\lambda_i\delta_{ij}$$

Therefore

$$p(h^{p-1}u'',\ y_j'')=-\lambda_j(h^{-1}u'',\ y_j\delta h)$$

Consequently

$$c_j = -p(h^{-1}u'',\ y_j\,\delta h)$$

From relation (4.1), using the last equality, we have

$$v_1''(x)h^p(x) = -p\sum\ (h^{-1}u'',\ y_i\delta h)y_i(x)$$

Substituting this expression into the first formula of (3.7) we obtain

$$\delta^2\Phi(h) = \chi^p(h, u, \delta h) + p^2\sum\ (h^{-1}u'',\ y_i\delta h)^2 \tag{4.2}$$

For the function $\psi(x)$, defined by the second equation of (3.5), the inclusion $\psi(x) \in L^2_{h,p}$ holds.

In fact, using the necessary conditions for an extremum (3.6) we obtain

$$\|\psi\|^2_{L^2_{h,p}} = \int_{-1}^{1} h^{-p}(x)|\psi(x)|^2\,dx = \int_{-1}^{1} h^{p-1}(x)(u''(x))^2 h^{-1}(x)\delta h^2(x)dx = l_0^2\int_{-1}^{1}\frac{\delta h^2(x)}{h(x)}dx\ .$$

By virtue of the assumptions made above regarding the order of degeneracy of the functions $h(x)$ and $\delta h(x)$ the last integral converges.

Substituting the function $\psi(x)$ into (4.2), we finally have

$$\delta^2\Phi(h) = -\frac{p(p-1)}{2}\|\psi\|^2_{L^2_{h,p}} + p^2\sum\ (\psi(x),\ y_i(x)h^{-p}(x))^2 \tag{4.3}$$

On the other hand $(\psi(x), y_ih^{-p}) = d_i$ are the Fourier coefficients in the expansion of the function $\psi(x)$ in eigenfunctions $\{y_i(x)\}$ of problem (2.2), (2.3) in the space $L^2_{h,p}$. Hence it follows that

$$\|\psi\|^2_{L^2_{h,p}} = \sum d_i^2 = \sum(\psi(x),\ y_ih^{-p})^2$$

Equations (4.3) now take the following form

$$\delta^2\Phi(h) = \left(p^2 - \frac{p(p-1)}{2}\right)\|\psi\|^2_{L^2_{h,p}} = \frac{p(p+1)}{2}\|\psi\|^2_{L^2_{h,p}} = \frac{p(p+1)}{2}l_0^2\int_{-1}^{1}\frac{\delta h^2(x)}{h(x)}dx \tag{4.4}$$

Consequently, $\delta^2\Phi(h)$ cannot be a negative-definite quantity for any permissible δh when $p > 0$ and $p < -1$ and a positive-definite quantity when $-1 < p < 0$. Taking the result of the previous theorem into account, we obtain the required assertion.

5. THE CASE OF RIGIDLY CLAMPED ENDS OF THE BEAM

It can be shown that the proof of the previous theorem can easily be extended to the case of boundary conditions (1.3). In fact, as will be shown below, the case of boundary conditions (1.3) is special in the sense that when $p > 1$ the optimum distributions of the thicknesses necessarily have zero values inside the range of the solution. It is interesting to note that it is precisely this case which is special in the well-known Lagrange problem on maximizing the stability of an elastic column [9].

Theorem 3. Suppose $f(x) \in W'$ and the necessary condition for an extremum (3.6) is satisfied for a certain $h(x) \in Q$. Then when $p > 1$ the functional $\Phi(h)$ may reach a local minimum only on the functions $h(x)$, which have zeros inside the solution range.

Proof. Suppose $p > 1$. Put $\delta h = u''(x)$ in formula (3.7). Using boundary conditions (1.3) it is easy to show that this variation satisfies the isoparametric condition (3.1). It follows from condition (3.6) that

$$|u''(x)| = l_0 h(x)^{(1-p)/2} \tag{5.1}$$

i.e. if, when $p > 1$, the function $h(x)$ does not vanish, the variation $\delta h(x) = u''(x)$ is a bounded function and, consequently, is permissible. Moreover, it can be shown that the function belongs to the space

$W^2_{h,p}$. In fact, taking (5.1) and the second condition of (1.4) into account, we have

$$\int_{-1}^{1} (u''(x))^2 h^p(x)dx = l_0^2 \int_{-1}^{1} h(x)dx = l_0^2$$

On the other hand, since condition (3.6) is satisfied, the second term on the right-hand side of (3.7) is equal to zero, since the function $v_1(x)$ satisfies boundary conditions (1.3). We have

$$(h^{p-1}u'',\ v_1''\delta h) = (h^{p-1}(u'')^2,\ v_1'') = l_0^2 \int_{-1}^{1} v_1''(x)dx = (v_1'(1) - v_1'(-1))l_0^2 = 0 \qquad (5.2)$$

Using this relation and condition (3.6), expression (3.7) for the second variation of the functional can be written in the form

$$\delta^2\Phi(h) = \chi^p(h, u, \delta h)$$

This expression is negative for any δh and $p > 1$.

Remark. All the previous discussions remain exactly the same for the case of $p < 0$ also. In fact, by virtue of (5.1) the variation will be permissible for any $h \in Q$.

6. ANALYTICAL AND NUMERICAL SOLUTIONS IN THE CASE OF A COLUMN WITH CLAMPED ENDS

An analytical solution of problem (1.1), (1.3) for $p = 3$ and $f(x) = \delta(x)$ was obtained in [7]. The optimum distribution of the thicknesses of the column in this case has the form

$$h(x) = \frac{3}{4}|2|x|-1|^{1/2} \qquad (6.1)$$

The corresponding solution of boundary-value problem (1.1), (1.3) is given by the formula

$$u''(x) = \frac{16}{27}|2|x|-1|^{-1/2}\operatorname{sign}(2|x|-1) \qquad (6.2)$$

The function $h(x)$ is degenerate at points $x = \pm 1/2$. It can be shown by a direct check that the second variation of the functional in solution (6.1), (6.2) is positive, i.e. the sufficient conditions for an extremum are satisfied in this case.

We will consider the general case of arbitrary $p \neq 0$ and $f(x) = \delta(x)$.

It follows from (1.1) that

$$u''(x)h^p(x) = |x|/2 + c_1x + c_2$$

In view of the symmetry of the problem $c_1 = 0$ and $u'(0) = 0$. Since $u'(1) = 0$, a point $x_0, x_0 \in (0, 1)$ exists such that $u''(x_0) = 0$. Hence

$$u''(x)h^p(x) = (|x| - x_0)/2$$

It then follows from the necessary conditions for an extremum (3.6) that

$$ph^{-(1+p)}(x) = (|x|-x_0)^2/4 = l_0^2$$

Hence

$$h(x) = c_p\,|(|x|-x_0)/2|^{1+q}, \quad c_p = (p/l_0^2)^{(1+q)/2}$$

Here was have introduced the notation

$$q = (1-p)/(1+p)$$

We will obtain the point x_0. Since $u'(0) = u'(1) = 0$, we have

$$\int_0^1 u''(x)dx = 0$$

Hence, we have the equation

$$-\int_0^{x_0} (x_0 - x)^q dx + \int_{x_0}^1 (x - x_0)^q dx = 0$$

the solution of which is $x_0 = 1/2$.

The constant c_p is found from isoperimetric condition (1.4).

We finally obtain that the solution of the original problem has the form

$$h(x) = \frac{2+q}{2} |2|x|-1|^{1+q} \tag{6.3}$$

It is shown in Fig. 1 for $p = 1, 2, 3$.

Here and henceforth, by virtue of symmetry, the graphs of the function $h(x)$ are only shown for $x \in [0, 1]$.

Note that solution (6.3) only has meaning when $p > -1$, since otherwise the function $h(x)$ will have a pole at the points $x = \pm 1/2$.

It can be shown by a direct check that the second variation of the functional in solution (6.3) is positive, i.e. the sufficient conditions for an extremum are satisfied in this case.

Consider the case when the transverse force in Eq. (1.1) has the form of two point loads, applied symmetrically about the centre of the beam

$$f(x) = (\delta(x-\xi) + \delta(x+\xi))/2, \quad 0 < \xi < 1$$

We have

$$u''(x)h^p(x) = (|x-\xi| + |x+\xi| + c_1 x + c_2) \tag{6.4}$$

By virtue of the symmetry of the problem $c_1 = 0$. The first part of Eq. (6.4) may change sign at the point x_0 when $c_2 = 2x_0$, $\xi < x_0 < 1$. Proceeding in the same way as in the previous case, we obtain

$$h(x) = c_p |(|x-\xi| + |x+\xi| - 2x_0)/4|^{1+q} \tag{6.5}$$

$$u''(x) = c_p^{-p} |(|x-\xi| + |x+\xi| - 2x_0)/4|^q z \tag{6.6}$$

$$z = \operatorname{sign}(|x-\xi| + |x+\xi| - 2x_0|$$

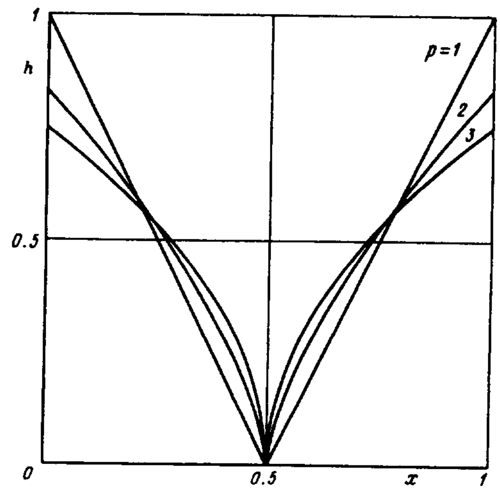

Fig. 1

To determine the point x_0 we have the equation

$$-\int_0^{\xi}(x_0-\xi)^q dx-\int_{\xi}^{x_0}(x-x_0)^q dx+\int_{x_0}^{1}(x-x_0)^q dx=0 \tag{6.7}$$

This equation does not have a solution for all ξ $(0 < \xi < 1)$.

In fact, suppose $p = 1$; then $x_0 = 1/2$. But a solution of Eq. (6.7) only exists when $\xi < x_0$. This means that when $\xi \geqslant 1/2$ no solution of the variational problem exists.

In Fig. 2 we show the behaviour of the function $h(x)$ for different values of ξ.

We will explain the meaning of this phenomenon using the example of the case $p = 3$, which allows of a clear mechanical interpretation. In this case, Eq. (6.7) takes the form

$$8x_0^2-4(2\xi+1)+\xi^2+4\xi=0 \tag{6.8}$$

and its real solutions are only possible when $0 < \xi < 1 + \sqrt{2}/2$.

In Fig. 3 we show a graph of x_0 as a function of ξ.

It can be seen from Eq. (6.8) that two values of x_0 correspond to each permissible value.

Graphs representing the optimum solutions for the first branch with $x_0 \in [(3-\sqrt{2})/4, 1/2]$ are shown in Fig. 4(a) for $p = 3$ and different values of ξ. The solutions for the second branch $x_0 \in [0, (3-\sqrt{2})/4]$, are shown in Fig. 4(b).

The values of the functional $\Phi(h)$ on each of these branches are equal to $c_p^{2/(1+q)}$.

We will calculate the value of the constant c_p. The function $h(x)$ is defined by expression (6.5). From the condition for the volume to be constant we obtain

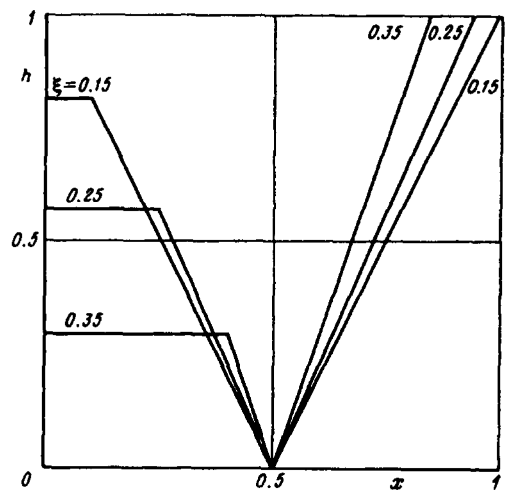

Fig. 2

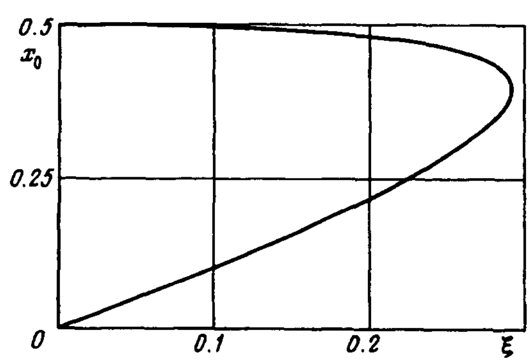

Fig. 3

$$c_p(\xi(x_0-\xi))^{1+q}+(2+q)^{-1}(x_0-\xi)^{2+q}+(2+q)^{-1}(1-x_0)^{2+q}=2^q$$

Calculations show that the value of the functional on the first branch $((3-\sqrt{2})/4 \leqslant x_0 \leqslant 1/2]$ will be less than on the second branch $(0 < x_0 < (3-\sqrt{2})/4)$ when $0 < \xi < 1-\sqrt{2}/2$.

If $\xi \geqslant 1-\sqrt{2}/2$, no solution of Eq. (6.8) exists. This means that $u''(x)$ does not change its sign when $|x| \leqslant 1$.

For an arbitrary value of p, the solution of the problem when $\xi \leqslant |x| \leqslant 1$ can be determined as follows:

$$h^p(x)u''(x)=(|x-\xi|+|x+\xi|-2\xi)/4$$

$$h(x)=c_p((|x|-\xi)/2)^{1+q}, \quad c_p=[(2+q)/4][(1-\xi)/2]^{-2-q}$$

The following consideration can serve as a mechanical justification of this solution: in this case the points of application of the external forces are so close to the clamped ends that, to minimize the deflection, a system of two columns is more rational. Hence, if the function $h(x)$ is supplemented to zero when $|x| < \xi$, the beam decomposes into two independent cantilevers, clamped at the points $x = \pm 1$.

The necessary condition for an extremum (3.6), like the original equation when $|x| < \xi$, formally ceases to be satisfied. At the points $x = \pm\xi$ we have the equations

$$(h^p u'')_{x=\pm\xi}=0, \quad (h^p u'')'_{x=\pm\xi}=\pm 1/2 \tag{6.9}$$

It can be shown that the optimum solution obtained corresponds to a cantilever if we directly consider the formulation of the initial problem for a cantilever beam. We have

$$(h^p(x)u'')''=\delta(x-\xi)/2, \quad h^p u''=|x-\xi|/4+c_1x+c_2$$

$$u(1)=u'(1)=0, \quad (h^p u'')_{x=0}=(h^p u'')'_{x=0}=0$$

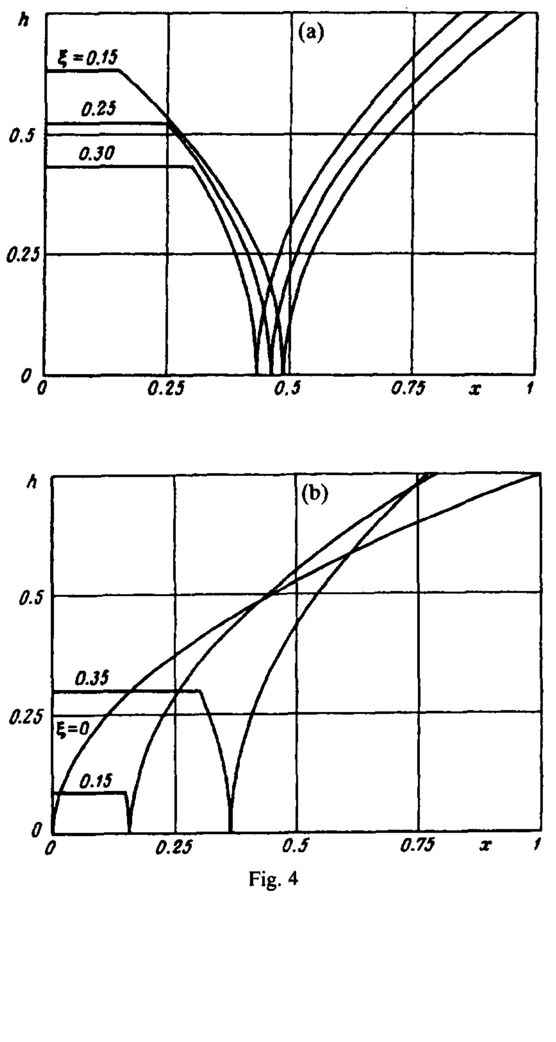

Fig. 4

We obtain from the boundary conditions $c_1 = 1/4$, $c_2 = -\xi/4$. Finally

$$h(x) = \begin{cases} 0, & x < \xi \\ c_p[(x-\xi)/2]^{1+q}, & x \geqslant \xi \end{cases}$$

It follows from isoperimetric condition (1.4) that

$$c_p = [(2+q)(1-\xi)^{-2-q}]/4$$

i.e. the solution agrees with the solution obtained previously in the case of clamped ends for $\xi \geqslant 1 - \sqrt{2}/2$.

To clarify the mathematical meaning of the solutions obtained above, we will consider the model problem of finding the solutions of the following boundary-value problem

$$(x^2y'(x))' = 0, \quad -1 < x < 1$$
$$y(-1) = 0, \quad y(1) = 0$$

The condition for the operator on the left-hand side of the equation to be positive-definite is not satisfied since the order of degeneracy at zero is equal to two, but it can be verified directly that the operator is positive.
Integrating the equation and using the boundary conditions we obtain

$$y(x) = -c_1/x + c_2, \quad c_1 = 0, \quad c_2 = 0$$

Nevertheless, the boundary-value problem has a non-trivial solution if we extend the range of the solutions to the set of generalized functions, namely, $y = \delta(x)$.
If the condition

$$\int_{-1}^{1} h^{-p}(x)dx < +\infty$$

is satisfied, the operator $A(h)$ is positive-definite, but, as the analytical solutions obtained above show, in the case of boundary conditions (2.4) this condition breaks down. Consequently, in this case homogeneous boundary-value problem (2.2), (2.4) can have non-trivial solutions, which makes the formalization of the optimization problem indeterminate.
All this shows that the solution of boundary-value problems (2.2)–(2.4) must be considered taking into account the order of degeneracy of the function $h(x)$, which was done in the initial formulation of the problem in Section 1. This condition does not limit the possibility of finding optimum solutions in the case of free supported ends, but, as the analytical solutions obtained show, it does considerably limit the possibility of obtaining optimum solutions in the case of clamped ends. The latter indicates that in this case one must reject the condition derived above regarding the order of degeneracy of the function $h(x)$.
We know from the general theory of boundary-value problems [10], that in this case, to ensure uniqueness of the solution of the boundary-value problem, we need to know additional boundary conditions at a point in the solution range, where the order of degeneracy is greater than $1/p$. For example, in the case of the boundary-value problem considered, $y(0) = 0$ serves as this condition. For the analytical solutions obtained above this condition changes into the first of conditions (6.9). Hence, the initial formulation of the problem for the case of clamped ends must contain this condition and the point in the interval where it must be satisfied is the unknown parameter of the problem and is determined during the course of the solution. This procedure was completely realized when seeking the analytical solutions given above.

From the practical point of view, it is important to investigate problems (1.1), (1.2) and (1.1), (1.3) with functional (1.5) and with limitations on the lowest value of $h(x)$

$$h(x) \geqslant h_{\min}, \quad h_{\min} > 0$$

By virtue of Theorem 3 this problem has no optimum solution, and hence we need to modify the original formulation of the initial problem appropriately. One of the ways of doing this is to introduce an additional limitation on the mean integral value of the square of the derivative of the function

$$\int_{-1}^{1} (h'(x))^2 dx \leqslant M, \quad M = \mathrm{const}$$

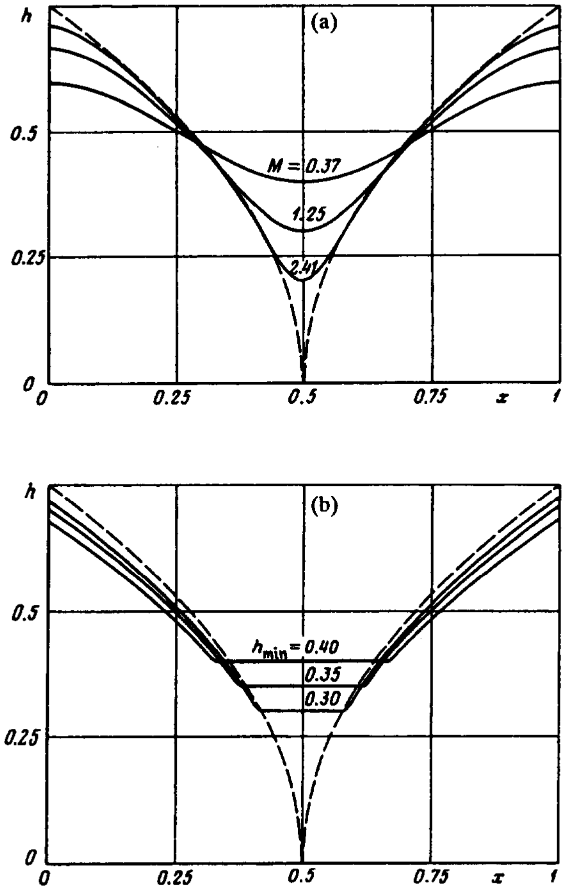

Fig. 5

Note that the value of the upper limit of the function $h(x)$ can be obtained from Poincare's inequality [3].

It was shown in [11] that in this case the optimum solution will exist if the constant is sufficiently small. Note that this condition is only sufficient for the existence of the optimum solution.

In Fig. 5(a) we show the result of a numerical solution of the modified problem with $f(x) = \delta(x)$, $p = 3$ and different values of the constant M. In Fig. 5(b) we show graphs of the solution of the problem with limits on the value of h_{min}, but without the integral limitation on the square of the product of the function $h(x)$, for different values of h_{min}.

As might have been expected, as M increases and h_{min} decreases the solutions of the modified problem tend to the optimum solution (6.1), shown by the dashed curve.

REFERENCES

1. KELLER, J. B., The shape of the strongest column. *Arch. Ration. Mech. and Analysis* 1960, **5**, 275–285.
2. KELLER, J. B. and NIORODSON, F. J., The tallest column. *J. Math. Mech.*, 1966, **16**, 433–446.
3. MIKHLIN, S. G., *Variational Methods in Mathematical Physics*. Nauka, Moscow, 1970.
4. KANMOROVICH, L. B. and AKULOV, L. P,. *Functional Analysis*. Nauka, Moscow, 1977.
5. PRAGER, W. and TAYLOR, J. E., Problems of optimal structural design. *Trans. ASME. Ser. EJ. Appl. Mech.*, 1968, **35**, 102–106.
6. TAYLOR, J. E., The strongest column: an energy approach. *Trans. ASME. Ser. E. J. Appl. Mech.*, 1967, **34**, 486–487.
7. BANICHUK, N. V. and KARIHALOW, B. L., On the solution of optimization problems with singularities. *Int. J. Solids and Structures*, 1977, **13**, 725–733.
8. KATO, T., *Perturbation Theory for Linear Operators*. Springer, Berlin, 1966.
9. BRATUS', A. S., The properties of the solutions of a class of isoperimetric problems of stability optimization. *Prikl. Mat. Mekh.*, 1994, **58**, 6, 89–95.
10. TRIBEL, H., *Interpolation Theory, Functional Spaces, Differential Operators*. Deutsch. Verl. Wissensch., Berlin, 1978.
11. BRATUS', A. S., The sufficient conditions for an extremum in problems of controlling the coefficients of elliptic operators. *Uspekhi Mat. Nauk*, 1985, **40**, 2, 171–172.

Translated by R.C.G.